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An elementary constructive approach to the higher-rank numerical ranges of unitary matrices and quantum error-correcting codes

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Abstract
One of the main ingredients of the quantum error-correcting codes theory is the study of the higher-rank numerical ranges of the operators related to the error operators. We constructively verify a conjecture on the structure of higher-rank numerical ranges for unitary matrices.

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1. Introduction

Quantum error correction is one of the main directions in the development of the quantum information theory since the middle of the 1990s, see [1–4]. The theory developed in these papers describes the noisy quantum channel as a completely positive, trace preserving map. This map acts on a quantum state ρ,

$$\rho \longrightarrow \sum_a E_a \rho E_a^+,$$

where the set of operators E_a satisfying condition $\sum_a E_a E_a^+ = I$ is interpreted as errors induced by environment. For some quantum channels the restriction of this map to a suitable subspace may have an inverse. Such subspaces are called quantum error-correcting codes (QECC). It was shown in this approach that if P is the projection related to QECC, then its existence is equivalent to the following relation:

$$P E_a^+ E_b P = \alpha_{ab} P \tag{1}$$

for any a, b and some constants α_{ab}. So, the finding of the QECC is reduced to the study of the matrix analysis problem. Namely, let $\mathbf{H}$ be a finite-dimensional Hilbert space, $\dim \mathbf{H} = N$. For $k \geqslant 1$ the *rank-k numerical range* of σ is the subset of complex plane $\Lambda_k(\sigma) = \{\lambda \in \mathbf{C} : P\sigma P = \lambda P$ for some k-dimensional orthogonal projections P on $\mathbf{H}\}$. Realization of the correctable quantum codes for the noisy quantum channel needs the explicit description of $\Lambda_k(\sigma)$ and the corresponding projections P for given operators σ which are related to errors operators in accordance with (1). Evidently, that $\Lambda_1(\sigma)$ is the operator spectrum $spec(\sigma)$. Recently, the higher-rank numerical range for unitary and normal matrices was studied in papers [5–7]. The next statement was proved in [7].

Proposition 1. *Let σ be a normal matrix acting on* **H** *and $k \geqslant 1$ be a positive integer, then*

$$\Lambda_k(\sigma) \subseteq \Omega_k(\sigma), \tag{2}$$

where

$$\Omega_k(\sigma) = \cap_\Gamma conv(\Gamma),$$

and Γ runs through all $(N - k + 1)$-point subsets (counting multiplicities) of the set of eigenvalues $spec(\sigma)$ for σ, $conv(\Gamma)$ means the convex hulls of the set Γ.

It was conjectured in [5] that the conversion of this statement is valid.

Conjecture. *For the normal matrix σ*

$$\Lambda_k(\sigma) = \Omega_k(\sigma). \tag{3}$$

For brevity we will denote the conjecture for given N, k as conjecture(N, k). This general statement was not proved in [7], where some particular cases were considered. In particular, the following propositions were proven in [7].

Proposition 2. *Conjecture (3) holds if and only if the corresponding statement holds for all unitary matrices.*

Proposition 3. *Conjecture(N, k) for $N \geqslant 3k, k \geqslant 2$, conjecture$(5, 2)$, conjecture$(8, 3)$, and conjecture$(3k - 1, k)$, $k \geqslant 2$ are valid.*

Conjecture(N, k), $N \geqslant 3k$, was verified in [7] explicitly. The corresponding construction was presented with the help of simple and elementary geometrical terms, see the discussion below. The verification of the conjecture $(5, 2)$, $(8, 3)$ and $(3k - 1, k)$ was given in [7] non-constructively. But for the realization of the quantum error-correcting codes it is necessary to get an explicit description of the corresponding objects, such as projection P. Recently, a full proof of the conjecture was obtained in [8] with the help of a more advanced technique.

The aim of this paper is to modify the elementary approach of the [7] and to suggest a constructive verification of the conjecture$(3k-1, k)$, $k \geqslant 2$, and conjecture$(3k-2, k)$, $k \geqslant 5$. We will consider here only the mathematical details, initial motivation and discussion of possible applications in the theory of quantum error-correcting codes can be found in [5–7].

2. General considerations

As it follows from proposition 2, we can discuss a unitary matrix σ, so its spectrum belongs to the unit circle. Let eigenvalues of σ be $\lambda_j = \exp(\mathrm{i}\theta_j)$, $j = 1, 2, \ldots, N$, such that $0 \leqslant \theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_N < 2\pi$. We extend the numbering of the λ_j and $|\psi_j\rangle$ cyclically if it is necessary. For multiple eigenvalues the numbering is arbitrary, and we choose an orthonormal system of eigenvectors $|\psi_j\rangle \in \mathbf{H}$,

$$\sigma|\psi_j\rangle = \lambda_j|\psi_j\rangle, \qquad j = 1, 2, \ldots, N. \tag{4}$$

We start with some conditions which guarantee the existence of the QECC. Let $\lambda \in \Lambda_k(\sigma)$, this means that for the corresponding k-dimensional orthogonal projection P

$$P\sigma P = \lambda P. \tag{5}$$

Let

$$P = \sum_{s=1}^{k} |\varphi_s\rangle\langle\varphi_s| \tag{6}$$

for some set of orthonormal vectors $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_k\rangle\}$ and

$$|\varphi_s\rangle = \sum_m z_{sm}|\psi_m\rangle. \tag{7}$$

Then the normalization condition means, that

$$\sum_m |z_{sm}|^2 = 1, \tag{8}$$

and orthogonality of different vectors $|\varphi_s\rangle, |\varphi_p\rangle$ means, that

$$\sum_m z_{sm}\overline{z_{pm}} = 0, \qquad s \neq p. \tag{9}$$

Relation (5) reads in our notations:

$$\sum_m \lambda_m |z_{sm}|^2 = \lambda, \qquad s = 1, 2, \ldots, k. \tag{10}$$

For the convenience of the following discussions we formulate the inversion of these considerations as a proposition.

Proposition 4. *If for given λ we can find a set of vectors $\{|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_k\rangle\}$ which satisfy relations (7)–(10), then $\lambda \in \Lambda_k(\sigma)$ and relation (6) describes the corresponding projection P.*

We need here one more result of [7] which will be useful in what follows.

Proposition 5. *Given integers i, j with $i < j < i + N$, let $D(i, j)$ denote the convex subset of $\mathbf{C}$ bounded by the line segment from λ_i to λ_j and the counterclockwise circular arc from λ_j to λ_i. Then*

$$\Omega_k(\sigma) = \cap_{i=1}^{N} D(i, i + k). \tag{11}$$

The next simple result which will be exploited below follows from proposition 5.

Corollary. *Let $T(i, j, m), i < j < m$, be a triangle on the complex plane with vertices $\lambda_i, \lambda_j, \lambda_m$. If $|j - i| < k$, $|m - j| < k$, $|N + i - m| < k$, then $\Omega_k(\sigma) \subset T(i, j, m)$, so for any $\lambda \in \Omega_k(\sigma)$ there exist nonnegative numbers p_i, p_j, p_m, for which the following relations are valid:*

$$p_i + p_j + p_m = 1, \tag{12}$$

$$\lambda_i p_i + \lambda_j p_j + \lambda_m p_m = \lambda. \tag{13}$$

Note that these relations are a special case of relations (8) and (10). We will associate with such a triangle a normalized vector

$$|\varphi_s\rangle = \sqrt{p_i}|\psi_i\rangle + \sqrt{p_j}|\psi_j\rangle + \sqrt{p_m}|\psi_m\rangle. \tag{14}$$

If we have k different such triangles, we obtain k different vectors (14), and if they are orthogonal to each other, relation (6) gives us corresponding QECC, $\lambda \in \Lambda_k(\sigma)$.

As was mentioned above, conjecture(N, k), $N \geqslant 3k$, was proved in [7]. Namely, the corresponding procedure includes a construction of k triangles satisfying conditions of the corollary. These triangles do not have common vertices (due to condition $N \geqslant 3k$), so the corresponding vectors (14) are orthogonal to each other. A full set of these k vectors satisfies proposition 4.

If $N = 3k - 1$, $N = 3k - 2$ we do not have such a set of triangles with different vertices and we need some modification of an elementary approach [7]. Namely, we also use a set of k triangles, but we permit the existence of the common vertex either for one pair of triangles (conjecture$(3k - 1, k)$), or for two pairs of triangles (conjecture$(3k - 2, k)$). The key result in our considerations is the following statement. It describes conditions, when two orthonormal vectors with necessary properties can be constructed with the help of two polygons (triangles) with common vertex.

Proposition 6. *Suppose we have two sets of numbers* $p_s, q_r \geqslant 0$, $s \in S \subset \{1, 2, \ldots, N\}$, $r \in R \subset \{1, 2, \ldots, N\}$, $S \cap R = \emptyset$, *which satisfy the following conditions:*

$$p_1 + \sum_{s\in S, s\neq 1} p_s = 1 \tag{15}$$

$$q_1 + \sum_{r\in R, r\neq 1} q_r = 1 \tag{16}$$

$$\lambda_1 p_1 + \sum_{s\in S, s\neq 1} \lambda_s p_s = \lambda, \tag{17}$$

$$\lambda_1 q_1 + \sum_{r\in R, r\neq 1} \lambda_r q_r = \lambda. \tag{18}$$

If either $p_1 \leqslant 1/2$ *or* $q_1 \leqslant 1/2$, *then there are two orthonormal vectors*

$$|\varphi_1\rangle = z_{11}|\psi_1\rangle + \sum_{s\in S\cup R, s\neq 1} z_{1s}|\psi_s\rangle, \tag{19}$$

$$|\varphi_2\rangle = z_{21}|\psi_1\rangle + \sum_{s\in S\cup R, s\neq 1} z_{2s}|\psi_s\rangle, \tag{20}$$

satisfying relations (8)–(10).

Remark. If S and R contain two numbers each, then relations (15)–(18) coincide with relations (12) and (13) respectively, and we can construct the corresponding pair of vectors (14). These vectors satisfy relations (8) and (10), but relation (9) fails. So we need more complex construction (19), (20) for orthogonality.

Proof. Let

$$\begin{aligned}|\varphi_1\rangle = {} & [\sqrt{p_1}\cos\theta + \mathrm{i}\sqrt{q_1}\sin\theta]|\psi_1\rangle \\ & + \exp(\mathrm{i}\alpha)\cos\theta \sum_{s\in S, s\neq 1} \sqrt{p_s}|\psi_s\rangle + \exp(\mathrm{i}\beta)\sin\theta \sum_{r\in R, r\neq 1} \sqrt{q_r}|\psi_r\rangle,\end{aligned} \tag{21}$$

$$|\varphi_2\rangle = [\sqrt{p_1}\cos\tau + \mathrm{i}\sqrt{q_1}\sin\tau]|\psi_1\rangle + \cos\tau \sum_{s\in S, s\neq 1} \sqrt{p_s}|\psi_s\rangle + \sin\tau \sum_{r\in R, r\neq 1} \sqrt{q_r}|\psi_r\rangle. \tag{22}$$

Simple calculations with the help of (15)–(18) confirm that relations (8) and (10) are valid for these vectors. We have to define values $\theta, \tau, \alpha, \beta$ in order to get orthogonality of vectors (21) and (22). This condition reads:

$$\begin{aligned}\langle\varphi_1, \varphi_2\rangle = {} & p_1\cos\theta\cos\tau + q_1\sin\theta\sin\tau + \exp(\mathrm{i}\alpha)\cos\theta\cos\tau(1 - p_1) \\ & + \exp(\mathrm{i}\beta)\sin\theta\sin\tau(1 - q_1) + \mathrm{i}\sqrt{p_1 q_1}(\sin\theta\cos\tau - \cos\theta\sin\tau) = 0.\end{aligned}$$

Here we take into account relations (15) and (16). Let $x = \tan\theta$, $y = \tan\tau$. Separating the real and imaginary parts of the last expression, one can get the following pair of equations for x, y:

$$p_1 + q_1 xy + \cos\alpha(1 - p_1) + \cos\beta(1 - q_1)xy = 0, \tag{23}$$

$$\sqrt{p_1 q_1}(x - y) + \sin\alpha(1 - p_1) + \sin\beta(1 - q_1)xy = 0. \tag{24}$$

Excluding y, we obtain the quadratic equation for x:

$$x^2 + Ax + B = 0, \tag{25}$$

where

$$A = \frac{(p_1 - 1)\sin\alpha[q_1 + (1 - q_1)\cos\beta] + (q_1 - 1)\sin\beta[(p_1 - 1)\cos\alpha - p_1]}{\sqrt{p_1 q_1}(q_1 + (1 - q_1)\cos\beta)}, \tag{26}$$

$$B = \frac{p_1 + (1 - p_1)\cos\alpha}{q_1 + (1 - q_1)\cos\beta}. \tag{27}$$

Equation (25) has a real root, when $A^2 \geqslant 4B$ or, in more detail,

$$\begin{aligned}&\{(p_1 - 1)\sin\alpha[q_1 + (1 - q_1)\cos\beta] + (q_1 - 1)\sin\beta[(p_1 - 1)\cos\alpha - p_1]\}^2\\ &\qquad \geqslant 4p_1q_1[p_1 + (1 - p_1)\cos\alpha][q_1 + (1 - q_1)\cos\beta].\end{aligned} \tag{28}$$

Note that if either $p_1 \leqslant 1/2$ or $q_1 \leqslant 1/2$ we can get the non-positive right-hand side of this expression by the corresponding choice of the parameters α, β. Then condition (28) holds, we can calculate corresponding (real) values x, y or, in other words, θ, τ and construct the pair of orthonormal vectors $|\varphi_1\rangle, |\varphi_2\rangle$ in explicit terms. The result follows. □

Definition. *Let* $p_s \geqslant 0, s \in S \subset \{1, 2, \ldots, N\}$ *and for given* λ

$$p_{s_1} + \sum_{s \in S \setminus s_1} p_s = 1 \tag{29}$$

$$\lambda_{s_1} p_{s_1} + \sum_{s \in S \setminus s_1} \lambda_s p_s = \lambda, \tag{30}$$

and $p_{s_1} \leqslant 1/2$. *We call the point* λ_{s_1} *the weak vertex of the polygon with vertices* $\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_m}, \{s_1, s_2, \ldots, s_m\} = S$.

In what follows we will construct k triangles $T(\lambda_i, \lambda_j, \lambda_m)$, each of them will satisfy conditions of the corollary. Note that each such triangle contains at least two weak vertices, see relation (12). For the conjecture $(3k - 1, k)$ only one pair of triangles (only two pairs for the conjecture $(3k-2, k)$) will have one common vertex, weak for one of the triangles. So, we can apply proposition 6 and get a pair of necessary vectors $|\varphi_1\rangle, |\varphi_2\rangle$ (two pairs for conjecture $(3k - 2, k)$, respectively). For the remaining $(k - 2)$ $((k - 4)$ for conjecture $(3k - 2, k))$ triangles the corresponding vectors will be defined by relation (14). These vectors will be normalized and orthogonal to each other, and the set of vectors will satisfy conditions (8)–(10). Then relation (6) gives us the corresponding QECC.

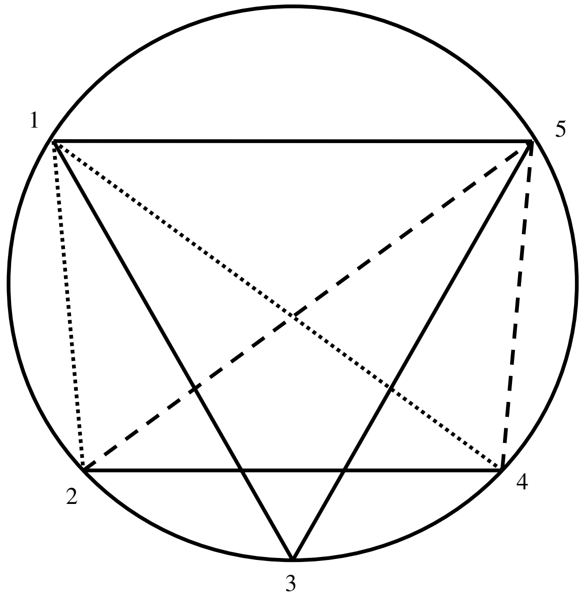

Figure 1. Conjecture (5, 2).

3. Constructive verification of the conjecture$(3k-1, k)$

So, our aim is to find a necessary system of triangles. In order to clarify details, we first consider $N = 5, k = 2$. The spectrum of the unitary operator σ is depicted in figure 1. Note that some eigenvalues can coincide, but we represent them as different points for more clearness. Let $\lambda \in \Omega_2(\sigma)$. In accordance with the corollary,

$$\lambda \in T(1,3,5) \cap T(1,2,4) \cap T(2,4,5).$$

As a first triangle, appearing in proposition 6, we take $T(1,3,5)$. Note that either vertex $\{1\}$ or vertex $\{5\}$ is the weak vertex of this triangle. In the first case we take $T(1,2,4)$ as a second triangle appearing in proposition 6. Then $\{1,3,5\} \cap \{1,2,4\} = \{1\}$. If the vertex $\{5\}$ is the weak vertex of triangle $T(1,3,5)$, we take $T(2,4,5)$ as the second triangle, $\{1,3,5\} \cap \{2,4,5\} = \{5\}$. In both situations chosen triangles contains only one common vertex, which is weak for triangle $T(1,3,5)$ and we can apply proposition 6. Then we obtain the pair of vectors satisfying relations (8)–(10). Two-dimensional QECC can be found with the help of relation (6).

As the second example we consider $N = 3k-1, k \geqslant 3$. The spectrum in this situation is depicted on figure 2. If $\lambda \in \Omega_k(\sigma)$, then, in accordance with the corollary, λ belongs to the triangle $T(1, k+1, 2k+1)$. In this triangle either vertex $\{1\}$ or $\{2k+1\}$ is the weak one. Let, for distinctness, it be $\{1\}$ (there is a symmetry of the picture). Then we take as the second triangle $T(1, k, 2k)$. The remaining $(k-2)$ triangles are $T(k-1, 2k-1, 3k-2)$, $T(k-2, 2k-2, 3k-3)$, etc. As follows from the corollary, $\Omega_k(\sigma)$ belongs to intersection of all triangles. Note that only triangles $T(1, k+1, 2k+1)$, $T(1, k, 2k)$ have one common vertex (which is weak for the first triangle). Applying proposition 6 to the triangles $T(1, k+1, 2k+1)$, $T(1, k, 2k)$, we can construct a pair of orthogonal vectors $|\varphi_k\rangle$, $|\varphi_{k-1}\rangle$, which satisfy conditions (8) and (10). For each triangle $T(k-m, 2k-m, 3k-m-1), m = 1, 2, \ldots, k-2$, we take, in accordance with (14), the vector

$$|\phi_m\rangle = \sqrt{p_{k-m}}|\psi_{k-m}\rangle + \sqrt{p_{2k-m}}|\psi_{2k-m}\rangle + \sqrt{p_{3k-m-1}}|\psi_{3k-m-1}\rangle,$$

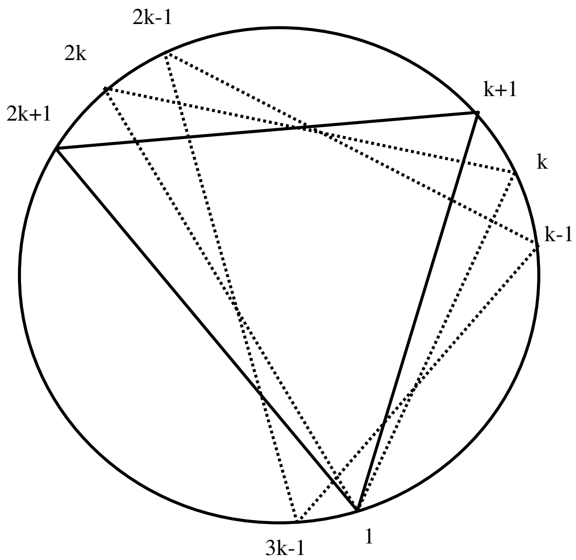

Figure 2. Conjecture $(3k-1, k)$, $k > 2$.

where the positive numbers p_{k-m}, p_{2k-m}, p_{3k-m-1} are determined by the relation

$$\lambda = \lambda_{k-m}p_{k-m} + \lambda_{2k-m}p_{2k-m} + \lambda_{3k-m-1}p_{3k-m-1}.$$

As was mentioned above, vectors $\{|\varphi_m\rangle, m = 1, 2, \ldots, k\}$ are orthogonal to each other. So, we have constructed the necessary set of vectors and the corresponding orthogonal projection is determined by relation (6).

4. Constructive verification of the conjecture$(3k-2, k)$

Now we consider conjecture$(3k-2, k)$, $k \geqslant 5$. In order to verify this situation we have to apply proposition 6 twice.

For the convenience we begin from conjecture(13, 5) (see figure 3). First we depict the triangle $T(1, 4, 9)$. In this triangle either $\{1\}$ or $\{4\}$ is the weak vertex. There is an evident symmetry of our figure at this stage, and we choose vertex $\{1\}$. Then the next triangle will be $T(1, 6, 11)$, which has one common vertex with triangle $T(1, 4, 9)$. With the help of proposition 6 we can construct two orthonormal vectors $|\varphi_1\rangle, |\varphi_2\rangle$, satisfying relations (8)–(10). As the next triangle we choose $T(3, 8, 12)$, where either $\{8\}$ or $\{12\}$ is its weak vertex.

(1) Let $\{8\}$ be the weak vertex of the triangle $T(3, 8, 12)$. Then we take triangle $T(5, 8, 13)$ and for the pair of triangles $T(3, 8, 12)$, $T(5, 8, 13)$ we apply proposition 6 and obtain the pair of orthogonal vectors $|\varphi_3\rangle, |\varphi_4\rangle$, satisfying relations (8)–(10). Residuary vertices give us the last triangle $T(2, 7, 10)$, which generates the fifth necessary vector in accordance with (14).

(2) Let $\{12\}$ be the weak vertex of the triangle $T(3, 8, 12)$. Then we choose triangle $T(2, 7, 12)$ and for the pair of triangles $T(3, 8, 12)$, $T(2, 7, 12)$ we apply proposition 6 to construct the pair of orthogonal vectors $|\varphi_3\rangle, |\varphi_4\rangle$, satisfying relations (8)–(10).

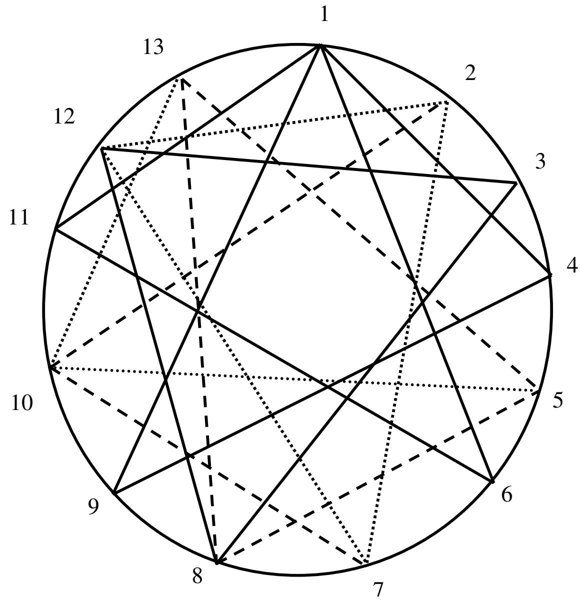

Figure 3. Conjecture (13, 5).

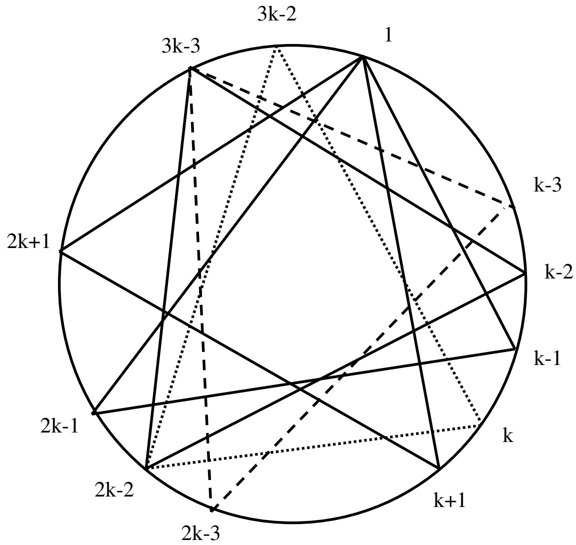

Figure 4. Conjecture $(3k-2, k)$, $k > 5$.

Residuary vertices give us the last triangle $T(5, 10, 13)$ and we obtain the fifth vector, associated with this triangle by (14).

Let us now consider conjecture $(3k-2, k)$ for $k > 5$, see figure 4. As the first triangle we choose triangle $T(1, k-1, 2k-1)$ and either $\{1\}$ or $\{k-1\}$ is its weak vertex. Due to symmetry we can choose anyone of them, and we choose $\{1\}$. The next triangle is $T(1, k+1, 2k+1)$, which has one common vertex with $T(1, k-1, 2k-1)$. So, proposition 6 gives the pair of vectors $|\varphi_1\rangle, |\varphi_2\rangle$ with necessary properties. Then we choose triangle $T(k-2, 2k-2, 3k-3)$. Here either $\{2k-2\}$ or $\{3k-3\}$ is its weak vertex.

(1) Let $\{2k-2\}$ be the weak vertex. Then we choose $T(k, 2k-2, 3k-2)$ as a next triangle. With the help of proposition 6 we construct one more pair of vectors $|\varphi_3\rangle, |\varphi_4\rangle$ with necessary properties. The triangle $T(2, k+2, 2k)$ gives the fifth vector $|\varphi_5\rangle$. Additional $(k-5)$ vectors can be constructed with the help of $k-5$ triangles $T(m, k+m, 2k+m-1)$, $m = 3, 4, \ldots, k-3$. Each such triangle satisfies the condition of the corollary and relation (14) gives us the corresponding vector.

(2) Now let $\{3k-3\}$ be the weak vertex. As a next triangle we choose $T(k-3, 2k-3, 3k-3)$. Applying proposition 6 we construct one more pair of vectors $|\varphi_3\rangle, |\varphi_4\rangle$ with necessary properties. The triangle $T(k, 2k, 3k-2)$ gives us the fifth vector $|\varphi_5\rangle$. Additional $(k-5)$ vectors can be constructed with the help of $(k-5)$ triangles $T(m, k+m, 2k+m-1)$, $m = 2, 3, \ldots, k-4$ and relation (14).

5. Conclusion

We have discussed the 'higher-rank numerical ranges' problem for constructing quantum error-correcting codes for quantum noisy channels. The realization of the correctable codes is reduced in this framework to the matrix analysis problem, which was thoroughly considered in papers [5–7] and solved in [8]. Realization of the error-correcting codes is based on explicit description of higher-rank numerical ranges of operators related to the error operators. The Corresponding constructive description was obtained in [7] for $N \geqslant 3k$ in elementary terms. Here we have presented some modification of this construction, which can be applied for $N = 3k-1, k \geqslant 2$ and $N = 3k-2, k \geqslant 5$. These results can be useful in constructing quantum error-correcting codes for special class of quantum channels. As we have considered unitary matrix problems, our results have a direct relation with binary unitary channels only (see the detailed discussion in [7]).

After the preliminary version of this paper [9] was finished, the author became aware of [8], where the same problem was considered from another point of view.

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